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A Difference between the Values of $|L(1/2+it,\chi_j)|$ and $|L(1/2+it,\chi_k)|$ II

by

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Abstract. Let $L(s,\chi)$ be the Dirichlet L function attached to a Dirichlet character χ. In this paper we are interested in the values of $|L(1/2+it,\chi_j)|^2-|L(1/2+it,\chi_k)|^2$, where χ_j and χ_k are primitive characters mod q and $\chi_j\neq\chi_k$. We study sign changes of $\int_0^T|L(1/2+it,\chi_j)|^2-|L(1/2+it,\chi_k)|^2dt$.

1. Introduction

Let $L(s,\chi)$ be the Dirichlet L function attached to a Dirichlet character χ. The asymptotic behavior of a integration

$$I(T,\chi)=\int_0^T|L(1/2+it,\chi)|^2dt \tag{1}$$

is a classical problem in analytic number theory. Let

$$M(T,q)=\frac{\phi(q)}{q}\left\{T\log T+T\left(\log\frac{q}{2\pi}+2\gamma-1+2\sum_{p|q}\frac{\log p}{p-1}\right)\right\}$$

where $\phi(q)$ is Euler's function and γ is Euler's constant. Here p runs over all prime divisors of q. Denote $I(T,\chi)-M(T,q)$ by $E(T,\chi)$, which would be the error term. In [3] we can see the result

$$E(T,\chi)=O\big((qT)^{1/3}(\log qT)^2+q^{1/2}(\log qT)^3\log T\big) \tag{2}$$

for $q=p$, a prime number.

Throughout this article χ_j and χ_k are primitive characters mod q. In [2] we studied the asymptotic behavior of

$$\int_0^T|L(1/2+it,\chi_j)-L(1/2+it,\chi_k)|^2dt \tag{3}$$

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and

$$\int_0^T |L(1/2+it,\chi_j)|^2 - |L(1/2+it,\chi_k)|^2 dt\,. \tag{4}$$

Using an approximate functional equation for L function, we showed that (3) is equal to

$$2\frac{\phi(q)}{q}T\log T + O_q(T\log^{3/4}T)$$

when $\chi_j \neq \chi_k$. While the behavior of (4) is more complicated than (3). Denote the integration (4) by $\Lambda(T)$. Because

$$\Lambda(T) = E(T,\chi_j) - E(T,\chi_k)\,,$$

the result (2) immediately leads to

$$\Lambda(T) = O\big((qT)^{1/3}(\log qT)^2 + q^{1/2}(\log qT)^3\log T\big)$$

for $q = p$, where p is a prime. In [2] we showed Atkinson type explicit formula for the error function $E(T,\chi)$. Using the formula we proved that

$$\Lambda(T) = \Omega(T^{1/4}) \tag{5}$$

for any fixed q, when $\chi_j \neq \chi_k$.

Our main problem in this paper is sign changes of $\Lambda(T)$ as $T\to\infty$ with fixed q.

THEOREM 1. *Assume that χ_j and χ_k are primitive characters mod q and $\chi_j \neq \chi_k$. Then there exist T_0, c_1, c_2 such that, for each $T > T_0$ there exist $t_1, t_2 \in [T, T+c_2\sqrt{T}]$ satisfying $\Lambda(t_1) > c_1 t_1^{1/4}$ and $\Lambda(t_2) < -c_1 t_2^{1/4}$.*

REMARK 1. Unfortunately we can not clear the dependency of T_0, c_1 and c_2 with respect to q.

CORORALLY 1. *Let χ_j and χ_k be as in Theorem 1. Then we have*

$$\Lambda(T) = \Omega_\pm(T^{1/4})\,.$$

In particular

$$\int_0^T |L(1/2+it,\chi)|^2 dt - \int_{-T}^0 |L(1/2+it,\chi)|^2 dt = \Omega_\pm(T^{1/4})$$

holds, when χ is a primitive complex character.

Our Theorem 1 is an analogue of D. R. Heath-Brown and K. Tsang's Theorem 1 of [1]. To prove our Theorem 1 we must study $E(T,\chi_j) - E(T,\chi_k)$ in detailed. Here we review an explicit formula of Atkinson type for $E(T,\chi)$.

2. On an explicit formula of Atkinson type for $E(T,\chi)$

Let

$$\begin{aligned}
e(T,u) &= \left(1+\frac{\pi u}{2T}\right)^{-1/4}\left(\sqrt{\frac{2T}{\pi u}}\operatorname{ar\,sinh}\sqrt{\frac{\pi u}{2T}}\right)^{-1},\\
f(T,u) &= 2T\operatorname{ar\,sinh}\sqrt{\frac{\pi u}{2T}}+\sqrt{2\pi uT+\pi^2u^2}-\pi/4\,,\\
g(T,u) &= T\log\frac{T}{2\pi u}-T+2\pi u+\frac{\pi}{4}\,,\\
l(T,u) &= \frac{T}{2\pi}+\frac{u}{2}-\sqrt{\frac{u^2}{4}+\frac{uT}{2\pi}}\,,\\
a(n,\chi) &= \frac{1}{q}\sum_{k|n}\sum_{a=1}^{q}\chi(a)\overline{\chi}(a+k)\exp\left(2\pi i\frac{a}{q}\frac{n}{k}\right),\\
b(n,\chi) &= \tau(\chi)^{-1}d(n)\chi(n)e^{2\pi in/q}\,,
\end{aligned}$$

where $\tau(\chi)$ is the Gauss sum and $d(n)$ is the divisor function. Then we have

(*Theorem 3 of* [2]) *Suppose that* $T\geq 10$ *and that* X *satisfies* $AqT<X<A'qT$ *for any fixed* $0<A<A'$. *If* χ *is a primitive character* mod q, *then we have*

$$E(T,\chi)=\Sigma_{1,\chi}\big(T,X\big)+\Sigma_{2,\chi}\big(T,l(T,X/q)\big)+R(T,\chi)\,, \tag{6}$$

where

$$\begin{aligned}
\Sigma_{1,\chi}(T,y) = q^{3/4}\Big(\frac{2T}{\pi}\Big)^{1/4}\sum_{n\leq y}\frac{|\overline{a(n,\overline{\chi})}|}{n^{3/4}}&\\
\times e(T,n/q)\cos\Big(f\big(T,n/q\big)-\pi n/q+\arg\overline{a(n,\overline{\chi})}\Big)&\,,
\end{aligned}$$

$$\Sigma_{2,\chi}(T,y)=-2q^{1/2}\sum_{n\leq qy}\frac{|\overline{b(n,\overline{\chi})}|}{n^{1/2}}\Big(\log\frac{Tq}{2\pi n}\Big)^{-1}\cos\Big(g(T,n/q)+\arg\overline{b(n,\overline{\chi})}\Big)$$

and $R(T,\chi)$ *is a certain quantity which satisfies*

$$R(T,\chi)\ll q(\log T)(\log^2 qT)+\frac{q^{3/2+\epsilon}}{\log^{1/6-\epsilon}T}\,.$$

This explicit formula plays a significant role in the proof of Lemma 1 stated in the next section.

3. The key Lemma

Let $f(t)$ be a real valued function satisfying $|f(t)| \le c_1 t^{1/4}$ for every large t. Here the constant c_1 is defined in the statement of Theorem 1. Define

$$\Lambda^*(t) = \frac{1}{\sqrt{2qt}}\Big(\Lambda(2\pi t^2/q) + f(2\pi t^2/q)\Big)$$

and

$$K_{\tau,\mu}(u) = \big(1-|u|\big)\left(1+\tau\sin\left(4\pi\theta\frac{\sqrt{\mu}}{q}u\right)\right)$$

for $|u| \le 1$, where $\tau = 1$ or -1, $\theta > 1$ is a large constant and $\mu \in \mathbb{N}$. In this section we prove the following:

LEMMA 1. *Let $\alpha(n) = a(n, \overline{\chi_j}) - a(n, \overline{\chi_k})$ and n_0 be a minimum of numbers n such that $\alpha(n) \neq 0$ for the given pair of characters. Suppose that $n_0 < t^{3/2}$, $2q \le t^2$ and $4\theta \le t$. For every large t, we have*

$$\begin{aligned}&\int_{-1}^{1} \Lambda^*(t+\theta u)K_{\tau,n_0}(u)du \\ &= \frac{|\alpha(n_0)|}{n_0^{3/4}}\left\{-\frac{\tau}{2}\sin\left(\frac{4\pi n_0^{1/2}}{q}t - \frac{\pi}{4} - \frac{\pi n_0}{q} - \arg\alpha(n_0)\right) + E\right\}\end{aligned} \tag{7}$$

where E is the error term estimated as

$$\begin{aligned}&\ll \frac{\theta^2 n_0^3}{q^2 t^4} + \frac{q^2}{\theta^2 n_0}\left(1+O\left(\frac{n_0}{t^2}\right)\right)^{-2} + \frac{n_0}{t^2} + \frac{n_0^{3/2}}{qt} \\ &\frac{n_0^{3/4}}{|\alpha(n_0)|}\left\{\frac{1}{t^{3/8-\epsilon}}\left(\frac{\theta^2}{q} + \frac{q}{\theta} + n_0^{1/2}\right)\right. \\ &+\frac{1}{t^{1/2-\epsilon}}\left(\frac{q^{1/2}}{\theta} + \frac{n_0^{1/2}}{q^{1/2}} + \frac{q}{t} + \frac{\theta}{q^{1/2}} + \theta + q^{1+\epsilon}\right) \\ &\left.+\frac{q^2}{\theta^2}\sum_{\substack{n\le t^{3/2}\\ n\neq n_0}} \frac{|\alpha(n)|}{n^{7/4}}\left\{1+\left(\left|1-\sqrt{\frac{n_0}{n}}\right| + O\left(\frac{n}{t^2}\right)\right)^{-2}\right\} + \frac{c_1}{q^{3/4}}\right\}.\end{aligned}$$

Proof. The structure of the proof is the same as that of Heath-Brown and Tsang's original argument [1], so we omit the details except for some crucial points.

Let $\eta = (3-\sqrt{5})/2$ and $\beta(n) = b(n, \overline{\chi_j}) - b(n, \overline{\chi_k})$. Using the explicit formula (6) with $T = 2\pi t^2/q$ and $X = t^2$, we have

$$\Lambda^*(t) = \Lambda_A(t) + \Lambda_B(t) + \frac{1}{\sqrt{2qt}}f(2\pi t^2/q) + O\left(q^{1/2}\frac{\log^3 t}{t^{1/2}} + q^{1+\epsilon}\frac{1}{t^{1/2}}\right)$$

where

$$\Lambda_A(t)=\sum_{n\le t^2}\frac{|\alpha(n)|}{n^{3/4}}\cos\biggl(f_A(t,n;q)-\frac{\pi n}{q}-\arg\alpha(n)\biggr)e_A(t,n)\,,$$

$$\Lambda_B(t)=-\frac{1}{\sqrt{2t}}\sum_{n\le\eta t^2}\frac{|\beta(n)|}{n^{1/2}}\left(\log\frac{t}{\sqrt{n}}\right)^{-1}\cos\bigl(g_B(t,n;q)-\arg\beta(n)\bigr)$$

with

$$\begin{aligned}
e_A(t,n)&=\left(1+\frac{n}{4t^2}\right)^{-1/4}\left(\frac{2t}{\sqrt{n}}\operatorname{arsinh}\frac{\sqrt{n}}{2t}\right)^{-1},\\
f_A(t,n;q)&=\frac{2\pi t\sqrt{n}}{q}\left(\frac{2t}{\sqrt{n}}\operatorname{arsinh}\frac{\sqrt{n}}{2t}+\left(1+\frac{n}{4t^2}\right)^{1/2}\right)-\frac{\pi}{4}\\
g_B(t,n;q)&=\frac{4\pi t^2}{q}\log\frac{t}{\sqrt{ne}}+\frac{\pi}{4}+\frac{2\pi n}{q}\,.
\end{aligned}$$

We see

$$\begin{aligned}
&\int_{-1}^{1}\Lambda^*(t+\theta u)K_{\tau,n_0}(u)du\\
&\quad=I_A+I_B+O\left(\frac{1}{\sqrt{qt}}\sup_{|u|\le1}\left|f\left(\frac{2\pi(t+\theta u)^2}{q}\right)\right|+q^{1/2}\frac{\log^3 t}{t^{1/2}}+q^{1+\epsilon}\frac{1}{t^{1/2}}\right),
\end{aligned}\tag{8}$$

where $I_j=\int_{-1}^{1}\Lambda_j(t+\theta u)K_{\tau,n_0}(u)du$ for $j=A,B$.

Our first task is to estimate I_B. Following Heath-Brown and Tsang (see p. 76 of [1]) and using the estimation $|\beta(n)|\le q^{-1/2}2d(n)$, we can replace the range of summation in I_B by ηt^2 with an error term $O(\theta q^{-1/2}t^{-1/2+\epsilon})$. Moreover using the first order estimation

$$\left(\sqrt{t+\theta u}\log\frac{t+\theta u}{\sqrt{n}}\right)^{-1}=\left(\sqrt{t}\log\frac{t}{\sqrt{n}}\right)^{-1}+O(\theta t^{-3/2})\,,$$

we have

$$I_B=-\frac{1}{\sqrt{2t}}\sum_{n\le\eta t^2}\frac{|\beta(n)|}{n^{1/2}}\left(\log\frac{t}{\sqrt{n}}\right)^{-1}J_B+O\left(\frac{\theta}{q^{1/2}t^{1/2-\epsilon}}\right),\tag{9}$$

where

$$J_B=\int_{-1}^{1}\cos\bigl(g_B(t+\theta u,n;q)-\arg\beta(n)\bigr)K_{\tau,n_0}(u)du.$$

Set $g(u)=g_B(t+\theta u,n;q)-\arg\beta(n)$. Integration by parts gives

$$J_B=\left[\frac{\sin\bigl(g(u)\bigr)}{g'(u)}K_{\tau,n_0}(u)\right]_{-1}^{1}-\int_{-1}^{1}\sin\bigl(g(u)\bigr)\frac{d}{du}\frac{K_{\tau,n_0}(u)}{g'(u)}du\,,$$

where the integral in the right-hand side is defined by $(\int_0^1 + \int_{-1}^0)\dots du$. Easy calculations show $g'(u) \gg q^{-1}\theta t$, $g''(u) \ll q^{-1}\theta^2 \log t$ and $K'_{\tau,n_0}(u) \ll q^{-1}n_0^{1/2}\theta$ for $|u| \le 1$ and $n \le \eta t^2$. By those estimations we have

$$J_B \ll \frac{q}{\theta t} + \frac{n_0^{1/2}}{t} + q\frac{\log t}{t^2}\,.$$

Combining this estimation of J_B and (9), we obtain

$$I_B \ll \frac{1}{q^{1/2}t^{1/2-\epsilon}}\left(\frac{q}{\theta} + n_0^{1/2} + \frac{q}{t} + \theta\right). \tag{10}$$

Next we consider I_A. As in the argument of I_B, we can replace the range of summation in I_A by t^2 with an error term $O(\theta t^{\epsilon-1/2})$. Moreover using the first approximation $e_A(t+\theta u, n) = e_A(t,n) + O(\theta t^{-1})$, we have

$$I_A = \sum_{n\le t^2} \frac{|\alpha(n)|}{n^{3/4}} e_A(t,n) J_A + O\left(\frac{\theta}{t^{1/2-\epsilon}}\right) \tag{11}$$

where

$$J_A = \int_{-1}^{1} \cos\left(f_A(t+\theta u, n; q) - \frac{\pi n}{q} - \arg\alpha(n)\right) K_{\tau,n_0}(u)du\,.$$

Set $f(u) = f_A(t+\theta u, n; q) - q^{-1}\pi n - \arg\alpha(n)$. By integrating by parts, we have

$$J_A = \left[\frac{\sin\big(f(u)\big)}{f'(u)} K_{\tau,n_0}(u)\right]_{-1}^{1} - \int_{-1}^{1} \sin\big(f(u)\big)\frac{d}{du}\frac{K_{\tau,n_0}(u)}{f'(u)}du\,,$$

where the integral in the right-hand side is defined by $(\int_0^1 + \int_{-1}^0)\dots du$. Using the estimations $f'(u) \gg q^{-1}\theta n^{1/2}$, $f''(u) \ll q^{-1}t^{-3}\theta^2 n^{3/2}$ and $K'_{\tau,n_0}(u) \ll q^{-1}n_0^{1/2}\theta$, we obtain

$$J_A \ll q\left(\frac{1}{\theta n^{1/2}} + \frac{\sqrt{n_0}}{q n^{1/2}} + \frac{1}{t^2}\right) \tag{12}$$

for $n \le t^2$.

Next we consider another estimation of J_A. In the sequel we denote $f_A(t,n;q)$ by f_A and $\lim_{x\to 0}\frac{d}{dx} f_A(t+x, n; q)$ by f'_A, respectively. We have $f''_A = O\big(q^{-1}n^{3/2}t^{-3}\big)$ and

$$f_A(t+\theta u, n; q) = f_A + f'_A\theta u + O\left(\frac{\theta^2 n^{3/2}}{qt^3}\right).$$

Hence it follows that

$$J_A = \int_{-1}^{1} \cos\big(C + f_A + f'_A\theta u\big) K_{\tau,n_0}(u)du + O\left(\frac{\theta^2 n^{3/2}}{qt^3}\right), \tag{13}$$

where $C=-q^{-1}\pi n-\arg\alpha(n)$. Straightforward calculations show

$$\begin{aligned}
&\int_{-1}^{1}\cos\bigl(C+f_A+f_A'\theta u\bigr)K_{\tau,n_0}(u)du\\
&\quad=2\int_0^1\bigl(\cos(C+f_A)\cos(f_A'\theta u)\bigr)(1-u)du\\
&\qquad-2\tau\int_0^1\bigl(\sin(C+f_A)\sin(f_A'\theta u)\bigr)(1-u)\sin\biggl(4\pi\theta\frac{\sqrt{n_0}}{q}u\biggr)du\\
&\quad=\cos\bigl(f_A+C\bigr)\biggl\{\frac{\sin(f_A'\theta/2)}{f_A'\theta/2}\biggr\}^2\\
&\qquad-\frac{\tau}{2}\sin\bigl(f_A+C\bigr)\biggl\{\frac{\sin\bigl((f_A'-4\pi\sqrt{n_0}q^{-1})\theta/2\bigr)}{(f_A'-4\pi\sqrt{n_0}q^{-1})\theta/2}\biggr\}^2\\
&\qquad+\frac{\tau}{2}\sin\bigl(f_A+C\bigr)\biggl\{\frac{\sin\bigl((f_A'+4\pi\sqrt{n_0}q^{-1})\theta/2\bigr)}{(f_A'+4\pi\sqrt{n_0}q^{-1})\theta/2}\biggr\}^2.
\end{aligned}$$

The right hand-side is equal to

$$\begin{aligned}
&-\frac{\tau}{2}\sin\Bigl(f_A(t,n_0;q)-\frac{\pi n_0}{q}-\arg\alpha(n_0)\Bigr)\\
&+O\biggl(\frac{q^2}{\theta^2 n_0}\frac{1}{\bigl(1+O(n_0/t^2)\bigr)^2}\biggr)+O\biggl(\frac{\theta^2 n_0^3}{q^2t^4}\biggr)
\end{aligned}\tag{14}$$

if $n=n_0$, and equal to

$$O\biggl(\frac{q^2}{\theta^2 n}\frac{1}{(1+O(n/t^2))^2}\biggr)+O\biggl(\frac{q^2}{\theta^2}\frac{1}{\bigl(n^{1/2}-n_0^{1/2}+O(n^{3/2}/t^2)\bigr)^2}\biggr)\tag{15}$$

if $n\neq n_0$. These estimations (14) and (15) are obtained by using

$$f_A'=\frac{8\pi t}{q}\,\mathrm{arsinh}\frac{\sqrt{n}}{2t}=\frac{4\pi\sqrt{n}}{q}+O\biggl(\frac{n^{3/2}}{qt^2}\biggr).$$

Applying (12) to the terms for $t^{3/2}<n\le t^2$ in (11), we have

$$\begin{aligned}
I_A&=\sum_{n\le t^{3/2}}\frac{|\alpha(n)|}{n^{3/4}}e_A(t,n)J_A+O\biggl(\frac{\theta}{t^{1/2-\epsilon}}\biggr)\\
&\quad+O\biggl(\sum_{t^{3/2}<n\le t^2}\frac{|\alpha(n)|}{n^{3/4}}\biggl(\frac{q}{\theta n^{1/2}}+\sqrt{\frac{n_0}{n}}+\frac{q}{t^2}\biggr)\biggr)\\
&=\sum_{n\le t^{3/2}}\frac{|\alpha(n)|}{n^{3/4}}e_A(t,n)J_A+r_1\,,
\end{aligned}\tag{16}$$

say. Here r_1 is estimated as

$$r_1\ll\frac{\theta}{t^{1/2-\epsilon}}+\frac{1}{t^{3/8-\epsilon}}\biggl(n_0^{1/2}+\frac{q}{\theta}\biggr)+\frac{q}{t^{3/2-\epsilon}}\,.$$

Next we substitute (13), (14), (15) and $e_A(t, n_0) = 1 + O(n_0/t^2)$ into the first sum in (16). Moreover using $f_A(t, n_0; q) = 4\pi\sqrt{n_0}tq^{-1} - \pi/4 + O\big(n_0^{3/2}(qt)^{-1}\big)$, we obtain

$$
\begin{aligned}
&\sum_{n \le t^{3/2}} \frac{|\alpha(n)|}{n^{3/4}} e_A(t, n) J_A \\
(17)\qquad &\qquad = -\frac{\tau}{2}\frac{|\alpha(n_0)|}{n_0^{3/4}} \sin\bigg(\frac{4\pi t\sqrt{n_0}}{q} - \frac{\pi}{4} - \frac{\pi n_0}{q} - \arg\alpha(n_0)\bigg) + r_2\,,
\end{aligned}
$$

where r_2 is an error term estimated as

$$
\begin{aligned}
&\ll \frac{1}{t^{3/8-\epsilon}}\frac{\theta^2}{q} + \frac{|\alpha(n_0)|}{n_0^{3/4}}\bigg(\frac{n_0^{3/2}}{qt} + \frac{\theta^2 n_0^3}{q^2 t^4} + \frac{q^2}{\theta^2 n_0}\frac{1}{\big(1 + O(n_0/t^2)\big)^2} + \frac{n_0}{t^2}\bigg) \\
&\quad + \frac{q^2}{\theta^2}\sum_{\substack{n \le t^{3/2} \\ n \ne n_0}} \frac{|\alpha(n)|}{n^{7/4}}\bigg\{1 + \bigg(\bigg|1 - \sqrt{\frac{n_0}{n}}\bigg| + O\Big(\frac{n}{t^2}\Big)\bigg)^{-2}\bigg\}.
\end{aligned}
$$

Combining (8), (10), (16) and (17), we complete the proof. □

4. Proof of Theorem 1

Let $\|x\|$ be the distance from x to the nearest integer. Suppose first that t satisfies

$$
(18)\qquad \bigg\|\frac{4\sqrt{n_0}}{q}t - \frac{1}{4} - \frac{n_0}{q} - \frac{\arg\alpha(n_0)}{\pi}\bigg\| \ge \frac{1}{6}\,;
$$

namely,

$$
\bigg|\sin\Big(\frac{4\pi\sqrt{n_0}}{q}t - \frac{\pi}{4} - \frac{\pi n_0}{q} - \arg\alpha(n_0)\Big)\bigg| \ge \frac{1}{2}\,.
$$

The right hand side of (7) is dominated by the first term, when we choose θ, t and c_1 suitably. We choose them as follows: c_1 is sufficiently small depending on n_0, θ is sufficiently large depending on q and n_0, and t is sufficiently large depending on q, n_0, c_1, and θ.

The right hand side of (7) takes both positive and negative values as τ changes sign. Hence $\Lambda^*(w)$ changes sign in the interval $[t - \theta, t + \theta]$ because $K_{\tau,n_0}(u)$ is non-negative.

Let S be a set consisting of real t satisfying (18). If sufficiently large t' is $\notin S$, we can find t'' such that $t'' \in S$ and $|t' - t''| < \frac{1}{12}\frac{q}{\sqrt{n_0}}$. Hence, $\Lambda(w)^*$ changes sign in

$$
\bigg[t - \theta - \frac{1}{12}\frac{q}{\sqrt{n_0}},\ \ t + \theta + \frac{1}{12}\frac{q}{\sqrt{n_0}}\bigg]
$$

irrespective of the validity of (18). Put $\sqrt{qT/(2\pi)} = t - \theta - \frac{1}{12}\frac{q}{\sqrt{n_0}}$ and $x = 2\pi w^2/q$. Then we can find

$$
x_1, x_2 \in \bigg[T,\ T + 4\sqrt{2\pi}\bigg(\frac{\theta}{q^{1/2}} + \frac{1}{12}\sqrt{\frac{q}{n_0}}\bigg)\sqrt{T} + 8\pi\bigg(\frac{\theta}{q^{1/2}} + \frac{1}{12}\sqrt{\frac{q}{n_0}}\bigg)^2\bigg]
$$

such that $\Lambda(x_1) + f(x_1) > 0$, $\Lambda(x_2) + f(x_2) < 0$.

We can apply this discussion to both the case $f(t) = c_1 t^{1/4}$ and $f(t) = -c_1 t^{1/4}$. Hence we obtain Theorem 1.

REMARK 2. In [2] we proved that the number n_0 exists for any fixed primitive characters with $\chi_j \neq \chi_k$. However we do not know the upper bound of n_0 and a lower bound of $|\alpha(n_0)|$ with respect to q. Thus we could not clarify the dependency of three numbers; T_0, c_1 and c_2, with respect to q.

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